similar but more pronounced effect of the variation in the value of Br_r on $\Lambda_s(\xi)$ was observed.

The unsteady state dimensionless temperature distribution $\Lambda_v(\xi, \tau)$ in the presence of viscous dissipation of heat can be obtained in the straight forward manner by using the substitution

$$U(\xi, \tau) = \Lambda_v(\xi, \tau) - \Lambda_p(\xi) \quad (14)$$

wherein $\Lambda_p(\xi)$ is given by equation (8) and solving the resulting equation by the method of separation of variables. The final expression for $\Lambda_v(\xi, \tau)$ when $pe/2$ is an integer can be shown to be

$$\Lambda_v(\xi, \tau) = \Lambda_s(\xi) - \sum_{m=1}^{\infty} \frac{\int_k^1 \Lambda_s(\xi)\xi^{1-pe/2} Z_{pe/2}(\lambda_m \xi) d\xi}{\int_k^1 \xi [Z^2_{pe/2}(\lambda_m \xi)] d\xi} \times \xi^{pe/2}[\exp(-\lambda_m^2 \tau/pr)] Z_{(pe/2)}(\lambda_m \xi) \quad (15)$$

wherein $Z_{(pe/2)}(\lambda_m \xi)$ and eigenvalues λ_m are defined in the same manner as the ones in case 1.

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ASYMPTOTIC SOLUTIONS FOR FORCED CONVECTION FROM A ROTATING DISK

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ROTATING disc systems are useful for precise heat and mass transport measurements. To analyze such measurements, an accurate expression for the Nusselt number $h\sqrt{(\nu/\omega)}/k$ (thermal) or $k_x\sqrt{(\nu/\omega)}/c\mathscr{D}_{AB}$ (binary) is required. The asymptotic formula of Levich [1],

$$Nu = 0{\cdot}620\, \Lambda^{\frac{1}{3}} \quad (1)$$

is suitable only when the Prandtl or Schmidt number, Λ, is very large: this formula overestimates Nu by 3 per cent even at $\Lambda = 1000$.

Gregory and Riddiford [2] have given a better expression,

$$Nu = \Lambda^{\frac{1}{3}}(1{\cdot}6126 + 0{\cdot}5705\, \Lambda^{-0{\cdot}36})^{-1} \quad (2)$$

for calculating Nu at moderately large Λ. This result is good within about 0·2 per cent for $\Lambda > 250$. Our purpose here is to give a more accurate result, valid down to $\Lambda \sim 1$, and also a new asymptote for $\Lambda \ll 1$. Our results are extensions of those given by Newman [7, 8] for $\Lambda \gg 1$ and $\Lambda \ll 1$, which came to our attention after this work was completed.

ANALYSIS

The formal solution for the Nusselt number on a rotating disc, in laminar flow with constant physical properties, is

$$Nu = \frac{1}{J(\Lambda)} \quad (3)$$

where

$$J(\Lambda) = \int_0^\infty \exp\left\{\Lambda \int_0^\zeta H(\zeta_1)\,\mathrm{d}\zeta_1\right\} \mathrm{d}\zeta. \qquad (4)$$

Here $\zeta = z\sqrt{\omega/\nu}$ is a dimensionless coordinate measured from the disk, and $H = v_z/\sqrt{\nu\omega}$ is a dimensionless velocity normal to the disk. The integration can be done numerically with available tables [3] of $H(\zeta_1)$; however, the following asymptotic formulas are more convenient.

ASYMPTOTE FOR LARGE Λ

If Λ is large, then the thermal or diffusional layer is thin, and $J(\Lambda)$ can be found analytically by expanding $H(\zeta_1)$ in powers of ζ_1. Using the values $H''(0) = -1{\cdot}02046$ and $G'(0) = -0{\cdot}6159$, reported by Sparrow and Gregg [4, 5], and calculating additional derivatives from these, we get:

$$\Lambda \int_0^\zeta H(\zeta_1)\,\mathrm{d}\zeta_1 = \Lambda \left\{-1{\cdot}02046\frac{\zeta^3}{3!} + 2{\cdot}00000\frac{\zeta^4}{4!} - 4\,(0{\cdot}6159)\frac{\zeta^5}{5!} + 1{\cdot}5173\frac{\zeta^6}{6!} + 2{\cdot}0409\frac{\zeta^7}{7!} + \ldots\right\}. \qquad (5)$$

Setting $x^3 = 1{\cdot}02046\,\Lambda\zeta^3/3!$ in this expansion, and inserting the result in equation (4), gives:

$$J(\Lambda) = \left(\frac{3!}{1{\cdot}02046\Lambda}\right)^{\frac{1}{3}} \int_0^\infty \exp(-x^3)\exp\begin{bmatrix} 0{\cdot}88435\,x^4\,\Lambda^{-\frac{1}{3}} \\ -\,0{\cdot}39323\,x^5\,\Lambda^{-\frac{2}{3}} \\ +\,0{\cdot}07285\,x^6\,\Lambda^{-1} \\ +\,0{\cdot}02527\,x^7\,\Lambda^{-\frac{4}{3}} \\ +\ldots \end{bmatrix}\mathrm{d}x. \qquad (6)$$

Expanding the second exponential in powers of Λ, and integrating each term as a gamma function, we obtain the following solution for Nu:

$$Nu = \Lambda^{\frac{1}{3}}/(1{\cdot}61173 + 0{\cdot}4803\,\Lambda^{-\frac{1}{3}} + 0{\cdot}2339\,\Lambda^{-\frac{2}{3}} + 0{\cdot}1132\,\Lambda^{-1} + 0{\cdot}05669\,\Lambda^{-\frac{4}{3}} + \ldots). \qquad (7)$$

The solution by Newman [7] corresponds to the first three terms of this series. From the $\Lambda^{-\frac{1}{3}}$ term on, this solution resembles a geometric series. Therefore, Aitken's extrapolation process [6] may be applied. Use of the first three terms of the series gives the following Aitken extrapolant.

$$Nu = \Lambda^{\frac{1}{3}}\Big/\left(1{\cdot}61173 + \frac{0{\cdot}4803}{\Lambda^{\frac{1}{3}} - 0{\cdot}4870}\right) \text{ for } \Lambda^{\frac{1}{3}} \gg 0{\cdot}4870 \qquad (8)$$

which has a series expansion very similar to equation (7).

Table 1 shows the relative accuracy of equations (1), (2), (7) and (8). Equation (8) is clearly the best.

Table 1. Comparison of results for $\Lambda \geq 1$

	$Nu\,\Lambda^{-\frac{1}{3}}$				
Λ	Equation (1)	Equation (2)	Equation (7) (5 terms)	Equation (8)	Exact value†
100	0·6205	0·5810	0·5789	0·5789	0·57892
	(7·2)*	(0·35)	(0·00)	(0·00)	
10	0·6205	0·5372	0·5266	0·5264	0·52640
	(17·9)	(2·04)	(0·04)	(0·00)	
5	0·6205	0·5175	0·4995	0·4989	0·49901
	(24·3)	(3·71)	(0·10)	(−0·02)	
2	0·6205	0·4861	0·4506	0·4478	0·44863
	(38·3)	(8·35)	(0·43)	(−0·19)	
1	0·6205	0·4581	0·4007	0·3925	0·39625
	(56·6)	(15·6)	(1·12)	(−0·95)	

* The numbers in parentheses are the per cent deviations of the predictions from the exact values.

† The values for $\Lambda = 2$ and 5 were calculated here: the others are from Sparrow and Gregg [4].

ASYMPTOTE FOR SMALL Λ

If Λ is small, then the thermal or diffusional boundary layer lies mostly outside the momentum boundary layer. For this region, the results of Cochran [3] and Sparrow and Gregg [4] can be fitted as follows:

$$\int_0^\zeta H(\zeta_1)\,\mathrm{d}\zeta_1 = H_\infty\zeta + C + D\,\mathrm{e}^{H_\infty\zeta} \qquad (\zeta \gg 1). \qquad (9)$$

We have recalculated the H function to obtain accurate values of the constants: $H_\infty = -0{\cdot}884464$, $C = 1{\cdot}6109$, and $D = -2{\cdot}365$.

Insertion of (9) into (4) gives:

$$J(\Lambda) = \exp(C\Lambda)\int_0^\infty \exp(\Lambda H_\infty\zeta)\exp(\Lambda D\,\mathrm{e}^{H_\infty\zeta})\,\mathrm{d}\zeta$$

$$= \exp(C\Lambda)\int_0^\infty \exp(\Lambda H_\infty\zeta)\sum_{m=0}^{\infty}\frac{(\Lambda D)^m}{m!}\exp(mH_\infty\zeta)\,\mathrm{d}\zeta$$

$$= \exp(C\Lambda)\sum_{m=0}^{\infty} -\frac{(\Lambda D)^m}{m!}\,\frac{1}{(\Lambda + m)H_\infty}. \qquad (10)$$

The Nusselt number thus becomes:

$$Nu = \frac{-H_\infty\Lambda\exp(-C\Lambda)}{\sum_{m=0}^{\infty}\frac{(\Lambda D)^m}{m!}\,\frac{\Lambda}{\Lambda + m}} \text{ for } \Lambda \ll 1. \qquad (11)$$

This result is accurate up to $\Lambda = 0{\cdot}1$, as shown in Table 2.

Table 2. Comparison of results for $\Lambda \leqslant 1$

Λ	Equation (12)	$Nu\,\Lambda^{-1}$ Equation (11)	Exact value [4]
0·01	0·88447 (1·60)*	0·87053 (0·00)	0·87051
0·1	0·88447 (11·5)	0·76842 (0·34)	0·76581
1·0	0·88447 (223·0)	0·46106 (16·4)	0·39625

* The numbers in parentheses are the per cent deviations of the predictions from the exact values.

The sum may be safely truncated to one term (unity) for $\Lambda < 0{\cdot}01$, and to three terms for $\Lambda < 0{\cdot}1$.

Sparrow and Gregg [4] have obtained the asymptote

$$Nu = 0{\cdot}88447\,\Lambda \text{ for } \Lambda \ll 1 \qquad (12)$$

by assuming a constant velocity $H = H_\infty$ for all $\zeta > 0$. This result is more restricted in range than equation (11), as shown in Table 2.

Newman has given the asymptote

$$Nu = 0{\cdot}88447\,\Lambda \exp(-1{\cdot}611\,\Lambda)(1 + 1{\cdot}961\,\Lambda^2) \quad \text{for } \Lambda \ll 1. \qquad (13)$$

This expression compares favorably with equation (11) near $\Lambda = 0{\cdot}1$, but is less accurate for larger and smaller Λ.

SUMMARY

Equations (8) and (11) cover the usual ranges of Λ for liquid-phase systems, including molten metals. Corrections for variable properties and net mass transfer will be given in forthcoming papers.

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